Directed percolation: a finite-size renormalisation group approach

1981 J. Phys. A: Math. Gen. 14 L163

(http://iopscience.iop.org/0305-4470/14/5/012)

J. Phys. A: Math. Gen. 14 (1981) L163–L168. Printed in Great Britain

LETTER TO THE EDITOR

Directed percolation: a finite-size renormalisation group approach

Wolfgang Kinzel† and Julia M Yeomans‡§

† Institut für Festkörperforschung der Kernforschungsanlage Jülich, 5170 Jülich, Postfach 1913, West Germany

‡ Baker Laboratory, Cornell University, Ithaca, NY 14853, USA

Received 11 February 1981

Abstract. The finite-size renormalisation group technique introduced by Nightingale is applied to the directed percolation problem. The decay of correlations is anisotropic in this model and finite-size scaling is extended to treat such anisotropy. Precise estimates for critical exponents and percolation probabilities are obtained for site, bond and site–bond percolation on the square lattice with bonds directed along the positive axes. Both free boundary conditions, for which the results converge linearly with $1/n$ as $n \to \infty$, and helical boundary conditions, for which, unexpectedly, the results converge linearly with $1/n^3$, are considered.

The percolation problem (see, for example, Stauffer (1979)), which describes lattices with sites or bonds present with probability p and absent with probability $1-p$, is of continuing interest. *Directed* (or oriented) percolation (Smythe and Wierman 1978) is, however, much less studied. The new feature which differentiates directed from ordinary percolation is a restriction on the direction of flow through the 'open' or 'occupied' bonds: for example, the flow of water through a porous medium becomes a directed percolation problem if gravity is important.

In this paper we study the critical properties of directed percolation on the square lattice illustrated in figure 1. Flow along occupied bonds and sites is permitted only along the direction of the arrows (in the forward 'time' direction). Bonds are present

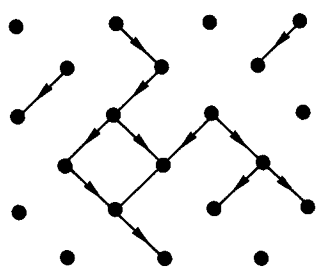

Figure 1. The oriented square lattice. Arrows indicate the allowed directions of flow along the bonds.

§ Address from September 1981: Department of Physics, The University, Highfield, Southampton SO9 5NH, England.

with probability p_b and sites with probability p_s, and bond percolation ($p_s = 1$), site percolation ($p_b = 1$) and bond–site percolation (p_s, p_b both variable) are considered.

We utilise the finite-size renormalisation group technique (Nightingale 1976). The directed percolation problem is particularly interesting because the decay of correlations is anisotropic. Specifically, if $\xi_\parallel$ and $\xi_\perp$ are the correlation lengths for decay in the flow direction and perpendicular to it, then we anticipate

$$\xi_\parallel \sim \xi_\perp^\theta \sim (p - p_c)^{-\nu_\parallel} \sim (p - p_c)^{-\nu_\perp \theta} \tag{1}$$

as the critical percolation probability, p_c, is approached. Here, θ is the anisotropy exponent and $\nu_\parallel$ and $\nu_\perp$ the correlation length exponents in the longitudinal and transverse directions respectively. A similar anisotropy of the correlation lengths occurs at a Lifshitz point (Hornreich *et al* 1975), which is currently being studied using similar methods (Yeomans, in preparation).

Both oriented and non-oriented percolation were first defined in the classic paper of Broadbent and Hammersley (1957). Series expansion results of Blease (1977a, b) and Monte Carlo work by Kertész and Vicsek (1980) and Dhar and Barma (1981) yielded estimates for the critical percolation probabilities on several oriented two-dimensional lattices, and indicated that directed and ordinary percolation are in different universality classes. Obukhov (1980) argued that $d = 5$ dimensions provided the upper critical dimension for the directed percolation problem (as opposed to $d = 6$ for ordinary percolation). He derived exponents to leading order in ε for $5 - \varepsilon$ dimensions. Recently Cardy and Sugar (1980) have shown that there is an exact mapping between the directed percolation problem and Reggeon field theory which models the creation, propagation and destruction of a cascade of elementary particles (Grassberger and de la Torre 1979).

Within the finite-size renormalisation group technique (Nightingale 1976, dos Santos and Sneddon 1980), a recursion relation is defined by considering the behaviour of the correlation length under a change in length scale. Thus if ξ_n is the correlation length of a strip of infinite length and of width n sites which is calculated by a transfer matrix technique, one defines the renormalised probability, p', via the relation

$$\xi_{n+1}(p') = [(n+1)/n]\xi_n(p). \tag{2}$$

The fixed point, p_n^*, and critical exponent, ν_n, are then defined as usual by

$$\xi_{n+1}(p_n^*) = [(n+1)/n]\xi_n(p_n^*), \tag{3}$$

$$\frac{1}{\nu_n} = \frac{\ln\left(\left.\frac{d\xi_n}{dp}\right|_{p_n^*} \Big/ \left.\frac{d\xi_{n+1}}{dp}\right|_{p_n^*}\right) - 1}{\ln (n/n+1)}. \tag{4}$$

For $n \to \infty$ one can show that $p_n^* \to p_c$ and $\nu_n \to \nu$ (dos Santos and Sneddon 1980). Note that (3) is equivalent to the assumption of finite-size scaling (Fisher 1972). Precise and apparently accurate values for critical exponents and transition parameters have been obtained using the finite-size renormalisation group for both thermal (Nightingale 1976, Sneddon 1978, dos Santos and Sneddon 1980, Ràcz 1980, Blöte *et al* 1980, Kinzel and Schick 1981) and geometrical (Derrida 1981, Derrida and Vannimenus 1981) models.

However, to treat the directed percolation problem, for which the correlation length is anisotropic, a modification of the approach is necessary. It is the *transverse* correlation length, $\xi_\perp$, which scales linearly with the width of the strip. However the *longitudinal* correlation length, $\xi_{\|}$, is calculated from the transfer matrix. Therefore from (1) and (3) we obtain

$$\frac{\xi_{\|,n+1}(p_n^*)}{\xi_{\|,n}(p_n^*)} = \left(\frac{\xi_{\perp,n+1}(p_n^*)}{\xi_{\perp,n}(p_n^*)}\right)^{\theta_n} = \left(\frac{n+1}{n}\right)^{\theta_n}. \tag{5}$$

This recursion relation contains two unknown parameters p_n^* and θ_n. These may be calculated by a comparison of strips of *three* successive widths, say $n-1$, n and $n+1$.

The transfer matrix technique used to calculate the correlation length in the oriented percolation problem is analogous to that used to treat ordinary percolation by Derrida and Vannimenus (1981). The matrix elements $\langle\psi_N|\mathbf{T}_n|\psi_{N+1}\rangle$ of the transfer matrix $\mathbf{T}_n$ are defined as the probability that row $N+1$ of the lattice is in a configuration $|\psi_{N+1}\rangle$ given that row N is in a configuration $|\psi_N\rangle$ where $|\psi_N\rangle$ describes whether or not the sites in row N are connected to row 1. A simplifying feature of directed percolation (compared with ordinary percolation) is that it is unnecessary to consider paths between sites in rows N and $N+1$ which proceed via earlier rows. The correlation length, ξ_N, is then given by

$$\xi_N = 1/(\ln \lambda_0^{(n)}/\ln \lambda_1^{(n)}) \tag{6}$$

(Camp and Fisher 1972) where $\lambda_0^{(n)}$, $\lambda_1^{(n)}$ are the two largest eigenvalues of $\mathbf{T}_n$. For the percolation problem, as a consequence of the normalisation of the probability, one has $\lambda_0^{(n)} \equiv 1$ and the contribution of the corresponding eigenvector can be factored out of the transfer matrix. The problem is therefore reduced to calculating a single largest eigenvalue, $\lambda_1^{(n)}$.

To calculate $\lambda_1^{(n)}$ a direct iteration technique has been used. Sparse transfer matrices (Domb 1949), which at each step add a single site, prove convenient in speeding the iterations and easing computer storage problems. If the lattice is built up by continued application of a sparse transfer matrix, helical boundary conditions result as illustrated in figure 2. We have studied both helical and free boundary conditions: in the latter case a different sparse matrix must be used to add the edge sites, which have a different connectivity from the sites in the bulk of the strip. For a strip of width n, the sparse transfer matrix is of size $2^n \times 2^n$ and contains 2^{n+1} non-zero elements. The largest strips considered were of width $n = 15$; the calculation took approximately 45 minutes on an IBM 370/168 machine.

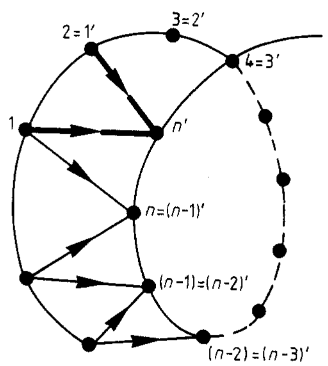

Figure 2. Construction of a strip of width n with helical boundary conditions using a sparse transfer matrix. Each iteration of the matrix adds a site n' together with the two bonds shown by bold lines and sums over the states of site 1.

Table 1 shows results for the site problem on the directed square lattice from calculations on strips of width $n-1$, n and $n+1$ with free boundary conditions. Values are given for p_n^*, the fixed point of the recursion relations, and the corresponding anisotropy and transverse correlation length exponents, θ_n and $\nu_{\perp n}$, obtained for values of n from 4 to 14. Extrapolations calculated from each successive pair of results, assuming that the results tend to a limiting value linearly with $1/n$, are also listed. The extrapolations suggest best values p_c (site) $= 0.7058 \pm 1$, $\theta = 1.581 \pm 1$, $\nu_\perp = 1.094 \pm 1$ where the errors refer to the final significant figure. Similar results were obtained for the bond problem and for the site–bond problem with $p = p_s = p_b$. The extrapolated values of the critical percolation probability and the critical exponents are listed in table 2.

Table 1. p_n^*, θ and $\nu_\perp$ for directed site percolation from calculations on strips of width $n-1$, n, $n+1$ with free boundary conditions. (The extrapolations are calculated from each successive pair of results, assuming that the results tend to a limiting value linearly with $1/n$.)

n	p_n^* (site)	Extrapolated values	θ_n	Extrapolated values	$\nu_{\perp n}$	Extrapolated values
4	0.703 78		1.5389		1.1313	
5	0.703 76	0.703 68	1.5387	1.5379	1.1278	1.1138
6	0.704 17	0.706 22	1.5439	1.5699	1.1230	1.0990
7	0.704 46	0.706 20	1.5483	1.5747	1.1193	1.0971
8	0.704 68	0.706 22	1.5519	1.5771	1.1162	1.0945
9	0.704 83	0.706 03	1.5549	1.5789	1.1138	1.0946
10	0.704 95	0.706 03	1.5574	1.5799	1.1119	1.0948
11	0.705 04	0.705 94	1.5596	1.5816	1.1102	1.0932
12	0.705 10	0.705 76	1.5614	1.5812	1.1089	1.0946
13	0.705 16	0.705 88	1.5629	1.5809	1.1077	1.0933
14	0.705 20	0.705 72	1.5642	1.5811	1.1067	1.0937
Best extrapolated value		0.7058 ± 1		1.581 ± 1		1.094 ± 1

We also studied the bond problem on the directed square lattice with helical boundary conditions. Unexpectedly the results converged linearly with $1/n^3$: intuitively a linear dependence on $1/n^2$, as exhibited by finite systems with periodic boundary conditions (Barber and Fisher 1973), seems more plausible. Extrapolated values are shown in table 2.

The values of p_n^* apparently converge very rapidly for $n \geqslant 9$, suggesting that the extrapolated values give rather accurate estimates for the critical percolation probabilities. An estimate of the accuracy of the extrapolated values of the critical exponents may be obtained by invoking universality. The values of θ obtained in the four different cases are identical (to within the error bars), which is very encouraging. The values of $\nu_\perp$ differ in the last figure, however, suggesting that final values should be $\nu_\perp = 1.098 \pm 5$, $\theta = 1.582 \pm 1$.

Results obtained by previous authors are listed in table 2 for comparison. Our results are in excellent agreement with those quoted by Cardy and Sugar (1980). The results also agree with the series work of Blease (1977a, b) and the Monte Carlo results of Dhar and Barma (1981), but lie outside the error bars of the values of Kertész and

Table 2. Extrapolated results for the critical percolation probability, p_c, the anisotropy exponent, θ, and the transverse and longitudinal correlation length exponents $\nu_\perp$ and $\nu_\parallel$. The results of the present work are compared with those of previous work on directed percolation and branching Markov processes (BMP).

	Boundary	p_c	θ	$\nu_\perp$	$\nu_\parallel$
Site	Free	0.7058 ± 1	1.581 ± 1	1.094 ± 1	1.730 ± 2
Bond–site	Free	0.8228 ± 1	1.582 ± 1	1.095 ± 2	1.732 ± 3
Bond	Free	0.644 ± 1	1.582 ± 1	1.099 ± 1	1.739 ± 2
Bond	Helical	0.6447 ± 1	1.582 ± 1	1.103 ± 1	1.745 ± 2
Previous results:					
Bond, Monte Carlo[a]		0.632 ± 4			1.65 ± 6
Bond, Monte Carlo[b]		0.6445 ± 5			
Bond, series[c]		0.6446 ± 2			1.730 ± 9
BMP, series[d]			1.572 ± 9		1.736 ± 1
BMP, Monte Carlo[e]			1.583 ± 10		1.691 ± 18

[a] Kertész and Vicsek (1980).
[b] Dhar and Barma (1981).
[c] Blease (1977).
[d] Brower *et al* (1978).
[e] Grassberger and de la Torre (1979).

Vicsek (1980), which were also obtained using Monte Carlo simulations. This may be a consequence of the difficulty of applying finite-size scaling arguments to a Monte Carlo simulation on a finite lattice when the decay of correlations in the system is anisotropic.

Finally, figure 3 shows the phase boundary for site–bond percolation on the directed square lattice. The curve plotted is obtained by extrapolation of the results for finite strips.

In conclusion, we have shown that the finite-size renormalisation group can be extended to treat systems with anisotropic correlation lengths. Precise estimates for the correlation length and anisotropy exponents and for the critical percolation probabilities have been obtained for bond, site and site–bond percolation on the oriented square lattice. Results obtained using helical boundary conditions were found to converge linearly with $1/n^3$. This is an unexpected result which deserves further investigation.

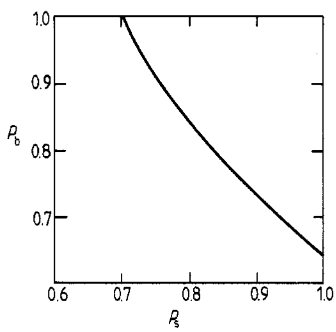

Figure 3. Critical percolation probability for site–bond percolation extrapolated from results obtained for strips of finite width.

It is a pleasure to thank J Kertész, T Vicsek and J L Cardy for interesting discussions, P Reynolds and S Redner for helpful correspondence and M E Fisher for a critical reading of the manuscript and for pointing out the linear dependence on $1/n^3$ of the results obtained using helical boundary conditions.

Part of this work was carried out at the Summer Institute in Theoretical Physics with partial support from the National Science Foundation under Grant No DMR-SC-06328 and the M J Murdoch Charitable Trust. One of us (JMY) acknowledges the support of the National Science Foundation through the Materials Science Centre at Cornell University.

References

Barber M N and Fisher M E 1973 *Ann. Phys.* **77** 1
Blease J 1977a *J. Phys. C: Solid State Phys.* **10** 917
——1977b *J. Phys C: Solid State Phys.* **10** 3461
Blöte H W J, Nightingale M P and Derrida B 1980 *Preprint*
Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629
Brower R, Furman M A and Moshe M 1978 *Phys. Lett.* **76B** 213
Camp W J and Fisher M E 1972 *Phys. Rev.* B **6** 946
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Derrida B 1981 *J. Phys. A: Math. Gen.* **14** L5
Derrida B and Vannimenus J 1981 *J. Phys. Lett.* **41** in press
Dhar D and Barma M 1981 *J. Phys. A: Math. Gen.* **14** L1
Domb C 1949 *Proc. R. Soc.* A **196** 36
Fisher M E 1972 *Proc.* 1970 *Enrico Fermi Summer School Course no* 51, *Varenna, Italy* ed. M S Green (New York: Academic)
Grassberger P and de la Torre A 1979 *Ann. Phys.* **122** 373
Hornreich R M, Luban M and Shtrikman S 1975 *Phys. Rev. Lett.* **35** 1678
Kertész J and Vicsek T 1980 *J. Phys. C: Solid State Phys.* **13** L343
Kinzel W and Schick M 1981 *Phys. Rev.* B in press
Nightingale M P 1976 *Physica* A **83** 561
Obukhov S P 1980 *Physica* **101A** 145
Ràcz Z 1980 *Phys. Rev.* B **21** 4012
dos Santos R and Sneddon L 1980 *Preprint*
Smythe R T and Wierman J C 1978 *Lecture Notes in Mathematics* **671** p 159
Sneddon L 1978 *J. Phys. C: Solid State Phys.* **11** 2823
Stauffer D 1979 *Phys. Rep.* **54** 1